SECTION 9.3 SUMMARY: SMOOTH-POINT INTEGRALS VIA RESIDUE FORMS
PRESENTED BY BRAD

1. Residues

As we've noted before, not all critical points contribute to our asymptotics. We will use residues to try to get the important ones, generalizing the univariate residue:

$$\mathrm{Res}(f, a) = \lim_{z \to a} (z - a) f(z),$$

for simple poles.

Definition 1. *Let F be a d-form on a domain in $\mathbb{C}^d$. Assume that F has a simple pole in a neighbourhood $U \subset V$, where V is the singular variety of F, and let i^* be the inclusion map from V to $\mathbb{C}^d$. Then we define*

$$Res(F, U) := i^* \theta,$$

where θ is a $(d-1)$-form satisfying $dh \wedge \theta = G dz$.

From the Appendix of the text, we know that θ is well-defined, and so is $i^*\theta$. The point of defining this residue is the following theorem:

Theorem 2 (9.3.2)**.** *Let*

- *$F = G/H$, with simple poles*
- *c be a real number such that V is smooth above the height $c - \epsilon$*
- *T be a torus on the boundary of a poly disc small enough to avoid V*
- *Y denote $M^{c-\epsilon}$ for some $\epsilon > 0$*

Then for all $\epsilon' < \epsilon$,

$$\left| a_{\underline{r}} - \frac{1}{(2\pi i)^{d-1}} \int_{INT[T,Y;V]} Res(w) \right| = O\left(e^{(c-\epsilon)|r|} \right),$$

where $INT[T, Y; V]$ is defined, as in appendix A.4., as the intersection of V and a $(d+1)$-chain with boundary in $T - Y$.

Proof. This follows from theorem A.5.3. □

Now with w as the Cauchy d-form

$$w = \underline{z}^{\underline{r}-1} \frac{G}{H} d\underline{z},$$

we shall compute the residue.

Proposition 3. *On a domain V where*

$$\frac{\partial G}{\partial z_1} \neq 0,$$

we have

$$Res(w) = \frac{z^{-\underline{r}-1} G}{\partial H / \partial z_1} dz_2 \wedge \cdots \wedge dz_d.$$

Furthermore, if

$$\frac{\partial G}{\partial z_k} \neq 0,$$

then

$$Res(w) = (-1)^{k-1} \frac{z^{-\underline{r}-1} G}{\partial H / \partial z_k} dz_1 \wedge \cdots \wedge dz_{k-1} \wedge dz_{k+1} \wedge \cdots \wedge dz_d.$$

Proof. We prove the first result, the second is obtained by flipping terms. For this, it is sufficient to show $dH \wedge \theta = \underline{z}^{\underline{r}-1} G d\underline{z}$, which follows from

$$\left(\sum_{j=1}^{d} H_j dz_j \right) \wedge \left(\frac{z^{-\underline{r}-1} G}{\partial H / \partial z_1} dz_2 \wedge \cdots \wedge dz_d \right) = H_1 dz_1 \wedge \frac{\underline{z}^{-\underline{r}-1} G}{H_1} \wedge dz_2 \wedge \cdots \wedge dz_d$$
$$= \underline{z}^{\underline{r}-1} G d\underline{z}.$$

□

Suppose $d = 2$. Then $w = x^{-r-1} y^{-s-1} \frac{G(x,y)}{H(x,y)} dx dy$. If $H_x \neq 0$ then

$$\text{Res}(w) = x^{-r-1} y^{-s-1} \frac{G}{H_x} dy,$$

and if $H_y \neq 0$ then

$$\text{Res}(w) = -x^{-r-1} y^{-s-1} \frac{G}{H_y} dx.$$

Example 1 (Binomial Coefficients)**.** *As seen before, we have $F = 1/(1-x-y)$, so that $G = 1$ and $H = 1-x-y$. This implies $H_x = -1 \neq 0$, and thus*

$$Res(w) = x^{-r-1} y^{-s-1} dy.$$

Example 2 (Delanoy Numbers)**.** *Now, we have $F = 1/(1-x-y-xy)$, so that $G = 1$ and $H = 1-x-y-xy$. This implies $H_x = -1-y$, and thus*

$$Res(w) = -x^{-r-1} y^{-s-1} \frac{1}{1+y} dy.$$

Using that $1-x-y-xy = 0$ on V, we see $-x-xy = y-1$ so that

$$Res(w) = x^{-r-1} y^{-s-1} \frac{x}{y-1} dy.$$

2. Return to Fourier-Laplace Integrals

Now, what we want to do is get these into the Fourier-Laplace integral form

$$\int_C e^{-\lambda \phi(\underline{z})} A(\underline{z}) d\underline{z}.$$

From Section 8.6 we know that the $\underline{z}^{-\underline{r}-1}$ factor can be pulled out of the residue – i.e.,

$$\text{Res}(w) = z^{-\underline{r}} \text{Res} \left(\underline{z}^{-1} F(\underline{z}) d\underline{z} \right).$$

Lemma 4 (9.3.2). *For*

$$C = \sum_{z \in W} C_*(\underline{z})$$

in $H_d(M, M^{c_*+\epsilon})$*, as in Lemma 8.2.4. in the text, we have*

$$\sigma = \sum_{\underline{z} \in W} C(\underline{z})$$

in $H_d(M, M^{c-\epsilon})$*, where*

(a) $\sigma = INT[T, Y; V]$ *is the intersection from Theorem 9.3.2.*

(b) W *is the set of critical points for* n *such that* $c_*(\underline{z})$ *is non-vanishing (see Lemma 8.2.4.)*

(c) $c_*(\underline{z})$ *is the relative cycle in definition 8.5.4. and is supported on* $M(\underline{z})$*, the union of* $M^{c-\epsilon}$ *with a small neighbourhood of* $\underline{z}$*.*

Thus, suppose again that $F = G/H$ has simple poles on V and is smooth above height $c - \epsilon$, and let N be a set of "nice" critical points, all of which are non-degenerate.

From Lemma 9.3.6., the fact that $\text{Res}(w) = z^{-\underline{r}}\text{Res}\left(\underline{z}^{-1}F(\underline{z})d\underline{z}\right)$, and Theorem 9.3.2. we get

$$\begin{aligned}
a_{\underline{r}} &= \frac{1}{(2\pi i)^{d-1}} \int_\sigma \text{Res}(w) + O\left(e^{(c-\epsilon)|r|}\right) \\
&= \frac{1}{(2\pi i)^{d-1}} \sum_{z \in \underline{W}} \int_C (z)\text{Res}(w) + O\left(e^{(c-\epsilon)|r|}\right) \\
&= \frac{1}{(2\pi i)^{d-1}} \sum_{z \in \underline{W}} \int_C (z)\text{Res}(w) + O\left(e^{(c-\epsilon)|r|}\right) \\
&= \frac{1}{(2\pi i)^{d-1}} \sum_{z \in \underline{W}} \int_C (z) z^{-\underline{r}}\text{Res}\left(\underline{z}^{-1}\frac{G}{H}d\underline{z}\right) + O\left(e^{(c-\epsilon)|r|}\right).
\end{aligned}$$

For convenience, we define $D_k := dz_1 \wedge \cdots \wedge dz_{k-1} \wedge dz_{k+1} \wedge \cdots \wedge dz_d$. Let $\phi(\theta) = \frac{-r}{|r|}\log(\underline{\theta})$, and $A(\underline{z})D_k = \text{Res}\left(\underline{z}^{-1}\frac{G}{H}d\underline{z}\right)$. We note that

$$\text{Res}\left(\underline{z}^{-1}\frac{G}{H}d\underline{z}\right) = z^{-1}\frac{G}{\partial H/\partial z_k}D_k,$$

so using Saddle Point methods we previously developed gives

Theorem 5 (9.3.7.).

$$a_r \sim \frac{|\underline{r}|^{(1-d)/2}}{(2\pi i)^{(d-1)/2}} \sum_{\underline{z} \in W} z^{-\underline{r}} \left(\det \mathcal{H}\left(\frac{r}{|r|}\right)\right)^{1/2} \frac{G(\underline{z})}{z_k H_k(\underline{z})}.$$